Existence and Estimation of Negative Critical Allometry Model Parameter

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Abstract- In this paper, we provide an interval of existence of negative critical mortality rate parameters M_r and b in Allometry survival model, in the absence of age-speci c mortality data by age.

Index Terms- Allometry, critical age, exponential growth, linear growth, mortality rate, , negative critical age.

I.INTRODUCTION

It has been recognized for more than 100 years that the rate of physiological processes is affected by body mass. Several theories have been put forth to explain the scaling of whole animal metabolic rate, ranging from single-cause explanations of supply system limitations (e.g., West et al. 2002)[1], to multi-cause assessments of the cellular pathways that determine the metabolic phenotype (e.g., Darveau et al. 2002)[2]. These model-oriented approaches address ultimate causes of scaling patterns across diverse models. An alter-native approach to understanding metabolic scaling is to address more prox-imate relationships. Muscle, for example, shows the same patterns of scaling of oxidative enzymes. Regardless of the underlying basis for metabolic phe-notype in whole animals, the molecular mechanism regulating the reciprocal change in muscle phenotype is still unknown. Muscles also exhibit phenotypic plasticity in bioenergetic enzymes as a result of di erences between muscle ber types, in response to ecological and behavioral changes and throughout ontogeny. It seems intuitive that within a species, individuals with larger bodies also have larger constituent parts. Larger humans tend to have longer legs, arms and torsos, bigger livers and larger hearts. This scaling relationship between the sizes of individual traits and the size of the whole body is called allometry[3]. Allometry describes how the characteristics of an organism scale with each other and with body size [4].

Accurately modelling the distribution of individual sizes at age is a fun-damental problem, which must be addressed when modelling the dynamics of a population. Many important characteristics of a population, such as mortality rate, are size speci c. For instance, shing mortality rate is depen-dent on the size-speci c selectivity of the shing gear. In most cases, natural mortality is also size dependent. In addition, for some species that cannot be aged, size is the only available measurement that provides information about reproductive maturity [5].

Manuscript Received on September 2013

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A theory of population that fails to consider a major determinant of the characteristics of populations is not an adequate theory. Standard texts in population biology and ecology tend to ignore body size as a factor in popu-lation dynamics, although birth and death rates, survivorship and longevity, population density and home range size, cycle periods for population boom and crash, and the annual increment in mortality due to aging all show a strong correlation with body mass (Calder [6]).Julian Huxley and Georges Teissier coined the term allometry in 1936. In a joint paper, simultaneously published in English and French (Huxley and Teissier [7]), they agreed to use this term in order to avoid confusion in the eld of relative growth. They also agreed on the symbols to be used in the algebraic formula of allometric growth:

$$Y = am^b$$

This makes body size a good choice for baseline analysis, using the scaling (heterogonic or allometric) equation of Huxley[8] and Kleiber[9]:

$$Y = am^b;$$

in which Y is a physiological, morphological, or ecological variable; the co-e cient a is characteristic of a class or order of animals and the physical dimensional units (if any) being used in the measurement of Y ; m is body mass (kg); and the exponent b is the ratio of changes in orders of magnitude for Y compared to m, thus expressing the e ect of body mass changes on Y.

The layout of this paper is as follows. First in section 2, we state the problem under consideration. This section is divided into two subsections, rst in subsection 2.1 fully discussed with the linear growth allometry model problem, under this assumption, we derived the negative critical allometry parameter. Second in subsection 2.2 deals with the exponential growth al-lometry model problem, under this condition we derived the corresponding negative critical allometry parameter. Finally we give the conclusion about this paper in section 3.

II. APPLICATIONS OF ALLOMETRIC SCALING LAW

A.Linear Growth

Stocking is widely used in the management of freshwater and, to a lesser extent, coastal-marine sheries(e.g., Heidinger[10]). A key problem in the management of stocked sheries is the optimization of release size (e.g., Cowx[11]). The optimal release size depends on the contribution that sh of a particular size will make to the catch or shable stock and on the re-sources required to

produce seed sh of that size. Of the data required to assess optimum size, the survival of seed sh of di erent sizes to the shable stock (and/or contribution to the catch) are the most di cult to obtain. Systematic assessments have been either entirely empirical (release - recap-ture of marked seed sh of di erent sizes) or based on detailed ecological studies(Wahl et al.[12]). However, the costs and e ort involved in both ap-proaches restrict their use to a small number of sheries, and the results are not readily generalized. An alternative approach that implies a simple gen-eralization is the use of allometric mortality-size relationship (Lorenzen[13]). Provided that natural mortality in stocked sh is subject to a consistent al-lometry, then an estimate of mortality for a single reference size is su cient to predict survival for a range of different release sizes.

Theoretical and empirical studies Peterson and Wroblewski[14], Loren-zen [15]) point to the existence of an allometric relationship between natural mortality and body weight, of the form

$$M_w = M_u W^{\,b} \qquad (1)$$

where M_w is natural mortality at weight W ; M_u is mortality at unit weight; b is the allometry exponent; and where there is an implied RHS coefficient of (unit weight)$^{-b}$. Note that a mathematical structure of this form would also apply to a system transformed to corresponding dimensionless variables as mentioned in the Introduction.

B.Survival Model

The survival model follows that developed by Lorenzen [16] in which the allometric relationship between natural mortality and body length may be described by the equation$_l$

$$M(l) = M_r\left(\frac{l}{l_r}\right)^b$$

where M(l) is the mortality rate at length l, M_r is the instantaneous mortality rate at reference length l_r (e.g., 15 cm { as used by Lorenzen [16]),and b is the allometric exponent of the mortality-length relationship. This reference length, l_r, needs to be chosen as a parameter such that it is smaller than another parameter, l_0, the length at stocking.

If this equation accurately describes mortality in the stocked population, then the decline in population size of a stocked cohort (organisms of the same age and size) of original population size N_0, while su ciently large enough to be approximated as a continuous variable, is described by the differential equation.

$$\frac{d}{dt}N(t) = -N(t)\, M_r \left(\frac{l(t)}{l_r}\right)^b \qquad (3)$$

where l(t) is length at time t and N(t) is the population size at time t. This differential equation may be solved explicitly if a linear growth model is substituted for l(t). A linear length growth model is reasonably used in the empirical analysis, because time at large (i.e., the time interval between release at stocking and estimated survival age at death or recapture) is short and the size of the sh is small relative to the reported maximum sizes in all stocking experiments analyzed in [16]. A model of the form

$$l(t) = lo + ut \qquad (4)$$

is used where t is the time since stocking, and u is the linear length growth rate. Substitution of equation (4) into equation(3), integration, and division by N_0 on both sides gives the following equation to predict survival, S(t) (proportion of stocked sh urviving), from the time of stocking to time t:

$$S(t) = \frac{N(t)}{N_0} = e^{-\frac{M_r\left(-l_0\left(\frac{l_0}{l_r}\right)^b + (l_0+ut)\left(\frac{l_0+ut}{l_r}\right)^b\right)}{(b+1)u}}, \qquad (5)$$

: as was derived by Lorenzen [16] for the case where $b \neq 1$.

The two parameters M_r and b are of interest to many investigators in biogerontology and the evolutionary biology of aging [17 - 21]. Species com-parisons in mortality rates are aided by calculations of MRD (mortality rate doubling time) which changes in the same direction as lifespan and is given by

$$\text{MRD} = \frac{2^{\frac{1}{b}} l_r - l_0}{u} \qquad (6)$$

In the presence of mortality data by age, the Allometric scaling param-eters M_r and b have been estimated by using various statistical methods like maximum likelihood, linear regression, and nonlinear regression[20-26]. Usually, an experimentalist knows the lifespan of each individual in a given population and can make use of standard techniques such as MLE or linear regression [22 & 23] to estimate the model parameters.

In the absence of age specific mortality data, in this paper we have de-veloped a method to estimate b from the instantaneous mortality rate at reference length, i.e., M_r; the original population size, N_0; and the maximum lifespan, t_m.

In [29] we have derived the following result for linear growth

$$t_m^* \approx t_m = \frac{1}{u}\left[(l_r)^{\frac{b}{b+1}}\left(\frac{(b+1)u\ln N_0}{M_r}\right) + l_o\left(\frac{l_0}{l_r}\right)^b\right)^{\frac{1}{b+1}} - l_0\right] \qquad (7)$$

The average mortality rate of steady state population subject to age specific mortality rates of equation (2) is [20 & 21]

$$A_{av} = \frac{1}{\int_0^\infty S(t)}dt\,. \qquad (8)$$

Equation (7) gives,

$$\frac{M_r}{(b+1)u} = \frac{\ln N_0}{-l_0\left(\frac{l_0}{l_r}\right)^b + (l_0+ut)\left(\frac{l_0+ut}{l_r}\right)^b} \qquad (9)$$

And from equation (8), we get

$$b+1 = A_{av}\, e^{x} \int_{x}^{\infty} \left(\frac{(b+1)l_r^{b}}{u^{b}}\right)^{\frac{1}{b+1}} \frac{e^{-z}}{z^{\frac{b}{b+1}}}\, dz, \qquad (10)$$

$$\approx A_{av}\, \frac{l_r}{u}\, e^{x} \int_{x}^{\infty} \frac{e^{-z}}{z}\, dz\,, \quad \text{see [29]}$$

Where $x = \frac{M_r l_0 \left(\frac{l_0}{l_r}\right)^{b}}{(b+1)u} = \frac{\ln N_0}{\left(\frac{l_0+ut_m}{t_0}\right)^{b+1-1}}$

C. Existence of negative critical parameter

Recall that the age at which the mortality rate, $M_r\left(\frac{l}{l_r}\right)^{b}$ of initial population N_0 has ceased increasing or, equivalently, it tends to a constant, is called a critical age, t_c [30]. The remaining population left from an original population size N_0 surviving at this critical age is called critical population, N_c, and the corresponding Allometry parameter in $M_r\left(\frac{l}{l_r}\right)^{b}$is called critical Allometry l_r parameter, b_c.Since the partials of b with respect to N_0 and t_m become zero at $M_r = \frac{A_{av} l_r^{b+1}}{l_0^{\,b}}$

and using (9), we get,

$$\frac{A_{av} t_m}{\ln N_0} = \frac{(b+1)ut_m}{l_0\left[\left(\frac{l_0+ut_m}{l_0}\right)^{b+1}-1\right]} \qquad (11)$$

From (11) it follows that

$$\frac{A_{av} t_m}{\ln N_0} = \begin{cases} < 1 \text{ if } b > 0, \\ = 1 \text{ if } b = 0, \\ > 1 \text{ if } b < 0. \end{cases} \qquad (12)$$

The point c at which b changes sign is said to be the critical point of b.

From our earlier work [30], we know that, for a given A_{av}, t_m ; and N_0

With $\frac{A_{av} t_m}{\ln N_0} < 1$.

$$t_c = t_m \quad \text{and} \quad N_c = e^{A_{av} t_c}. \qquad (13)$$

On the contrary, when $\frac{A_{av} t_m}{\ln N_0} > 1$,

$$t_c = \frac{\ln N_0}{A_{av}}, \quad N_c = e^{A_{av} t_c} \quad \text{and} \quad t_c \neq t_m \qquad (14)$$

Note that t_c t_m for any given A_{av}; t_m and N_0. We also know that [31], the asymptotic solution b of (10) is a continuous function in the variables A_{av}; t_m and N_0 from puberty through critical life span (or, N_0 N_c). What happens to solution b when N_0 N_c (below the critical population). As we have already equated t_m with t_c when $\frac{A_{av} t_m}{\ln N_0} < 1$, what is then actual (species) maximum life span, t_m?. Such $t_m > t_m$ exists, since the critical population N_c has not yet diminished to one survivor. How to determine this t_m? This in turn, leads us to consider negative allometry parameter, since $\frac{A_{av} t_m}{\ln N_0}$ 1

for $N_0 = N_c$.

To determine the negative allometry parameter consider (13) with N_0 N^c and (14) with t^c $\frac{\ln N_0}{}$. Upon substitution $N^0 = N^c$ into (10), we get

$$b_c + 1 = A_{av}\, e^{\xi} \int_{\xi}^{\infty} \frac{e^{-z}}{z}\, dz\,, \qquad (15)$$

With

$\xi = \frac{\ln N_c}{\left(\frac{l_0+ut_c}{t_0}\right)^{b_c+1-1}}$, where $t_c = t_m$ when $\frac{A_{av} t_m}{\ln N_0} < 1$.

And $t_c = \frac{\ln N_0}{A_{av}}$ when $\frac{A_{av} t_m}{\ln N_0} > 1$.

Eq. (15) gives

$$b_c + 1 \leq \frac{A_{av} t_c}{\ln N_c}\, l_0\left[\left(\frac{l_0+ut_c}{l_0}\right)^{b_c+1} - 1\right].$$

The above inequality ensures the existence of solution α_c, provided $\frac{A_{av} t_c}{\ln N_c} = 1$. As a consequence, we get

$$b_c + 1 \leq l_0\left[\left(\frac{l_0+ut_c}{l_0}\right)^{b_c+1} - 1\right]. \qquad (16)$$

A close observation of the inequality (16) reveals that any $b \leq 0$ also satisfies it. This result, in fact, motivates us to consider the negative critical allometry parameter. On the other hand, in the neighbourhood of the critical point $\frac{A_{av} t_c}{\ln N_c} = 1$, it is necessary that $\frac{A_{av} t_c}{\ln N_c} > 1$ if $b < 0$ from (12). Hence

$$(-b_c + 1)ut_c \leq l_0\left[\left(\frac{l_0+ut_c}{l_0}\right)^{b_c+1} - 1\right]. \qquad (17)$$

Invalid for $N_0 = N_c$. provided $\frac{A_{av} t_c}{\ln N_c} > 1$. Clearly $\frac{A_{av} t_c}{\ln N_c} > 1$ holds when $t_c > \frac{\ln N_0}{A_{av}}$ in (14). In case of (13,) $\frac{A_{av} t_c}{\ln N_c} > 1$ if $N_0 < N_c$ and this gives $t_c > \frac{\ln N_0}{A_{av}} = t_c \frac{\ln N_c}{\ln N_0}$. We designate $t_m^* = t_c \frac{\ln N_c}{\ln N_0}$ as the actual maximum life span.

Since t_m^* becomes very large numerically as $N_0 \to 1$ [30], we can choose any finite value greater than t_c and t_m^*. Fortunately, we shall not use the the numerical value of t_m in our sequel.

Numerical experiments show that the transition from b_c to b_c is ex-tremely slow. The graphical illustration should enhance the understanding of this idea [19]. Further, it follows from (9) that as $b \to 0$ at $N_0 = N_c$., $M_r \to A_{av}$, since

$$M_r = \frac{\ln N_c}{t_c} = A_{av}.$$

As b_c changes sign, from (9) we obtain

$$\frac{M_r}{(-b_c+1)u} = \frac{\ln N_c}{-l_0 \frac{l_0^{\,b_c}}{l_r^{\,b_c}} + (l_0+ut_c)\left(\frac{l_0+ut_c}{l_r}\right)^{-b_c}} \qquad (18)$$

Which gives

$$M_r = \frac{\ln N_c}{t_c} \frac{(-b_c+1)ut_c}{-l_0 \frac{l_0^{\,b_c}}{l_r^{\,b_c}} + (l_0+ut_c)\left(\frac{l_0+ut_c}{l_r}\right)^{-b_c}} \geq \frac{\ln N_c}{t_c}$$

Thus

$$M_r \geq \frac{\ln N_c}{t_c}.$$

C.Linear Allometry Survival function

In the neighbourhood of the critical point, $\frac{A_{av}t_c}{\ln N_c}$ =1 in view of (16) and (17) the survival function (5) takes the form

$$S(t)=\begin{cases} e^{\frac{-M_r-l_0\frac{l_0}{l_r}^{b}+(l_0+ut_c)(\frac{l_0+ut}{l_r})^{-b}}{(b+1)u}} & \text{if } t \leq t_c (N \geq N_c.) \quad (19) \\ e^{\frac{-M_r-l_0\frac{l_0}{l_r}^{-b}+(l_0+ut)(\frac{l_0+ut}{l_r})^{-b}}{(-b+1)u}} & \text{if } t \leq t_c (N \leq N_c.) \end{cases}$$

Lemma: S(t) is continuous at t = t_c.

Proof: Indeed

$$\lim_{t\to t_c-0} S(t) = e^{-\frac{M_{rc}\left(-l_0\frac{l_0}{l_r}^{-b_c}+(l_0+ut_c)\left(\frac{l_0+ut_c}{l_r}\right)^{-bc}\right)}{(b_c+1)u}}$$

$$= e^{\frac{-M_{rc}}{(b_c+1)u}(-l_0\frac{l_0}{l_r}^{-b_c}+(l_0+ut_c)(\frac{l_0+ut_c}{l_r})^{-bc})}$$

$$= e^{\frac{-M_{rc}}{(b_c+1)u}(-l_0\frac{l_0}{l_r}^{-b_c}+(l_0+ut_c)(\frac{l_0+ut_c}{l_r})^{-bc})}$$

$$= \lim_{t\to t_c-0} S(t)$$

Now we extend the formula (15) to -b_c for $N_0 = N_c$.
When given A_{av}, t_m and N_0 satisfy (13) with $N_0 \leq N_c$

And (14) with $t_c \leq \frac{\ln N_0}{A_{av}}$

Upon substitution $b_c = b_c$ into (15) we get

$$-b_c + 1 = A_{av}\frac{l_r}{u}\frac{-\ln N_c}{e^{1-(\frac{(l_0+ut_c)}{t_0})-b_c+1}}$$

$$\int_{\frac{-\ln N_c}{1-(\frac{(l_0+ut_c)}{t_0})-b_c+1}}^{0} \frac{e^{-z}}{z}\,dz$$

As such the above integral is unbounded. The unboundedness results in by the substitution $t_c = 1$ or $N_0 = 1$ with N_0 N_c into (15), when $b_c < 0$. But inequality (17) implies the existence of b_c.

To overcome this situation, we need to introduce limit age, t_{lim} of the critical population N_c.

Following Suematsu [32], the limit age can be de ned as an age where the nal member of the critical population N_c disappears. Stating mathematically,

$$S_{(tim)} = e^{\frac{-M_r-l_0\frac{l_0}{l_r}^{b}+(l_0+ut_{tim})(\frac{l_0+ut_{tim}}{l_r})^{-b}}{(-b+1)u}} \leq \frac{1}{N_0}$$

Clearly, if the population N_c at an age $t \leq t_c$ is less than unity, all the members under discussion must, in the statistical mean, vanish. Hence, the limit age can be identified with the minimum age, t_{lim}, that satisfies

$$S_{(tim)} \leq \frac{1}{N_0}$$

Taking into account the above arguments, substitute $N_0 = 0$ ($N_0 \leq N_c$) into the upper limit of (15) when $b_c < 0$ to get

$$-b_c + 1 = A_{av}\frac{l_r}{u}\frac{-\ln N_c}{e^{1-(\frac{(l_0+ut_c)}{t_0})-b_c+1}} \quad E_i\left(\frac{lnN_c}{1-\left(\frac{(l_0+ut_c)}{t_0}\right)^{-b_c+1}}\right)$$

(21)

where

$$E_i(x) = \lim_{n\to+0}\left[\int_{-x}^{-\eta}\frac{e^{-z}}{z}dz + \int_{\eta}^{\infty}\frac{e^{-z}}{z}dz\right]$$

[x>0]

Clearly, (21) satis es inequality (17). We could not obtain (17) directly from (21), since the integrand is unbounded at z = 0.

From equation (21), on account of 8.214. 2 in [33], we get

$$b_c + 1 = A_{av}\frac{l_r}{u}e^{-r}\left[C + \ln r + \sum_{k=1}^{\infty}\frac{r^k}{k\,k!}\right].$$

Where C=0.557215, Euler's constant and

$$\tau = \frac{lnN_c}{1-(\frac{(l_0+ut_c)}{t_0})-b_c+1}.$$

Next, we shall obtain a greatest lower bound for b_c.

From the above equation, we get

$$b_c + 1 \geq A_{av}\frac{l_r}{u}e^{-r}[C + \ln r + e^{-r} - 1],$$

Since

$$\sum_{k=0}^{\infty}\frac{x^k}{k\,k!} < \sum_{k=1}^{\infty}\frac{x^k}{k!} < \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x - 1.$$

Further

$$b_c + 1 \geq -A_{av}\frac{l_r}{u}e^{-\tau}[C+\ln\tau] - A_{av}\frac{l_r}{u}[1-e^{-\tau}]$$

$$\geq -A_{av}\frac{l_r}{u}e^{-r}[C+\ln\tau] - A_{av}\frac{l_r}{u}$$

$$\geq -A_{av}\frac{l_r}{u}[C+1] - A_{av}\frac{l_r}{u}\max(ln\tau, e^{-\tau})$$

$$= -A_{av}\frac{l_r}{u}[C+1] - 0.097A_{av}\frac{l_r}{u}$$

$$(0.097 = \max(ln\tau, e^{-\tau}))$$

$$= -A_{av}\frac{l_r}{u}[1.097 + C].$$

On the other hand, from the inequality (17) we get

$$(b_c + 1)ut_c \leq l_0\left[\left(\frac{l_0+ut_c}{l_0}\right)^{b_c+1} - 1\right]$$

Combininig , finally we get

$$-A_{av}t_c l_r[1.097 + C] \leq (b_c + 1)ut_c \leq l_0\left[\left(\frac{l_0+ut_c}{l_0}\right)^{b_c+1} - 1\right].$$

Theorem 1: For every fixed A_{av}; t_m and N_0 satisfying either (13) with $N_0 \leq N_c$ $t_c \geq \frac{\ln N}{A_{av}}$ or (14) with let I be the interval defi ned by

$I = (-A_{av}\frac{l_r}{u}[1.097 + C], 0)$. !

where C = 0:577215, Euler's constant

Suppose there exists a unique solution of (21) in I. Then it is necessary

that. $\frac{A_{av}t_c}{\ln N_c} = 1$

Moreover, the following estimation is true

$$b_c + 1 \geq -A_{av}\frac{l_r}{u}[1.097 + C].$$

Remark 1. From (18) it is easy to get the asymptotic formula of initial mortality rate using (21).

D.Exponential Growth

Two modes of growth have been proposed in the ecdysozoan: \saltational," in taxa in which a tanned cuticle permits size increase only at molts, and \con-tinuous" in taxa with stretchable, collagenous cuticles [34]. Research into these methods of growth has been limited almost exclusively to the arthro-pods (saltational growth) and nematodes (continuous growth), and even here, despite long standing interest in the details of the saltational growth of arthropod taxa (Alpatov [35], Rice [36]), continuous growth has rarely been investigated closely (Howells and Blainey [37], Wilson [38]). Speci caly, little is known of how continuous growth is achieved at a ne scale, the role of cuticle, and the cells that secret it. Understanding the details of growth has important implications for understanding the signi cance of molting as an evolutionary conserved feature of the ecdysozoa (Wilson [38]) and for inter-preting the increasing number of studies that seek to identify the molecular and cellular controls of the ecdysozoan growth (Estevez et al.[39], Johnston et al.[40], Oldham et al. [41]).

In [34], Knight et al. used the free living nematode Canenorhabditis elegans as the best characterized example of continuously growing ecdysozoan (Riddle et al.[42]). The hatchling worm is 0.25 mm long and grows to 1.4 mm within 5 days, a 6-fold increase in length and over a 100-fold increase in volume. C. elegans have an S-shaped growth curve an exponential phase of larval growth and a gradual approach to a plateau in late adulthood (Byerly et al.[43]). In view of this we assume that the growth variable l(t) (see equation (4)) is exponential. That is

$$l(t) = l_0 e^{vt} \quad (22)$$

where v is the allometric exponent.

E.Survival Model

A model of the form given in (22) is used. Substitution of equation (22) into equation(3), integration, and division by N_0 on both sides gives the following equation to predict survival, S(t),

$$S(t) = e^{-M_r(\frac{l_0}{l_r})^b \frac{e^{bvt}-1}{bv}} \quad (23)$$

The two parameters M_r and b are of interest to many investigators in biogerontology and the evolutionary biology of aging [17 - 21]. Species com-parisons in mortality rates are aided by calculations of MRD (mortality rate doubling time) which changes in the same direction as lifespan and is given by

$$MRD = \frac{1}{v}\ln\left(\frac{2^{\frac{1}{b}}l_r}{l_0}\right). \quad (24)$$

In the absence of age-speci c mortality data, we have developed a method to estimate b from the instantaneous mortality rate (M_r), original population size (N_0), and maximum lifespan (t_m).

In [29] we have derived the following result for exponential growth

$$t_m^* \approx t_m = \frac{1}{b_v}\ln[1 + b_v\frac{\ln N_0}{M_r}\left(\frac{l_r}{l_0}\right)^b]. \quad (25)$$

The average mortality rate of a steady state population subject to age speci c mortality rates of equation (19) is [20 & 21]

$$A_{av} = \frac{1}{\int_0^\infty S(t)dt}, \quad (26)$$

Equation (25) gives

$$\frac{M_r}{b_v}\left(\frac{l_0}{l_r}\right)^b = \frac{\ln N_0}{(e^{b_v t_m}-1)}, \quad (27)$$

And from equation (27) we get,

$$\frac{1}{A_{av}} = \int_0^\infty e^{-M_r(\frac{l_0}{l_r})^b \frac{e^{bvt}-1}{bv}}\,dt.$$

A simple substitution in the above integral gives

$$b(= \frac{A_{av}}{v}\frac{\ln N_0}{(e^{b_v t_m}-1)}\int_{\frac{\ln N_0}{(e^{b_v t_m}-1)}}^\infty \frac{e^{-\tau}}{\tau}d\tau. \quad (28)$$

F.Existence of Negative Critical Parameter

Recall that the age at which the mortality rate, $M_r(\frac{l}{l_r})^b$ of initial population N_0 has ceased increasing or, equivalently, it tends to a constant, is called a critical age, t_c [30]. The remaining population left from an original population size N_0 surviving at this critical age is called critical population, N_c, and the corresponding Allometry parameter in $M_r(\frac{l}{l_r})^b$ is called Allometry parameter, b_c.

Since the partials of b with respect to N_0 and t_m become zero at

$b = \frac{A_{av}}{v}(\frac{(e^{b_v t_m}-1)}{\ln N_0})$, and using (27), we get

$$\frac{A_{av}t_m}{\ln N_0} = \frac{b_v t_m}{(e^{b_v t_m}-1} \quad (29)$$

From (29) it follows that

$$\frac{A_{av}t_m}{lnN_0}\begin{cases} < 1 \; if \; b > 0, \\ = 1 \; if \; b = 0, \\ > 1 \; if \; b < 0. \end{cases} \qquad (30)$$

The point $\frac{A_{av}t_m}{lnN_0} = 1$ at which b changes sign is said to be critical point of b.

In human populations, according to published studies (Witten, [20 & 24]), the acceleration of mortality rate slows after 85 years. After 105 years, the mortality rate appears to cease increasing and may even decrease at these extremely advanced ages. Decreasing mortality at advanced ages is described in detail for ies (Curtsinger et al., 1992 [44], Fukui et al., 1993 [45], Carey et al., 1992 [46]). There is an additional evidence for the exponential decay at higher age. Quite recently Wang and co-workers disclosed an elegant experiment for the senescence accelerated mouse (SAM), showing that the mouse mortality function also approaches a constant value at higher age (Wang et al., 1998[47]). All the evidences accumulated so far suggest strongly that the exponential decay of populations at higher age is a general theorem (K.Suematsu et al., 1999 [32]).

From our earlier work [30], we know that, for a given N_0, t_m and A_{av} with $\frac{A_{av}t_m}{lnN_0} < 1$

$t_c = t_m$ and $N_c = e^{A_{av}t_c}$. (31)

On the contrary, when $\frac{A_{av}t_m}{lnN_0} > 1$

$$t_c = \frac{lnN_0}{A_{av}}, N_c = e^{A_{av}t_c} \quad \text{and} \quad t_c \neq t_m \qquad (32)$$

Note that t_c t_m for any given A_{av}; t_m and N_0. We also know that [31], the asymptotic solution b of (28) is a continuous function in the variables A_{av}; t_m and N_0 from puberty through critical life span (or, $N_0 \geq N_c$). What happens to solution b when $N_0 \leq N_c$ (below the critical population). As we have already equated t_m with t_c when, $\frac{A_{av}t_m}{lnN_0} < 1$,what is then actual (species) maximum life span, t_m^*?. Such $t_m^* < t_m$ exists, since the critical population N_c has not yet diminished to one survivor. How to determine this t_m^*? This in turn, leads us to consider negative allometry parameter, since $\frac{A_{av}t_m}{lnN_0} \geq 1$ for $N_0 \leq N_c$.

To determine the negative allometry parameter consider (31) with $N_0 \leq N_c$.and (32) with $t_c \geq \frac{\ln N_0}{A_{av}}$.

Upon substitution $N_0 = N_c$ into (28), we get

$$b_c = \frac{A_{av}}{v}\frac{\ln N_c}{(e^{b_c v t_c}-1)}\int_{\frac{\ln N_c}{(e^{b_c v t_c}-1)}}^{\infty}\frac{e^{-z}}{z}dz \qquad (33)$$

Where $t_c = t_m$ when $\frac{A_{av}t_m}{lnN_0} < 1$ and $t_c = \frac{lnN_0}{A_{av}}$ when $\frac{A_{av}t_m}{lnN_0} > 1$.

Equation (33) gives

$$b_c v t_c \leq \frac{A_{av}t_c}{lnN_c}(e^{b_c v t_c} - 1)$$

The above inequality ensures the existence of solution b_c, provided $\frac{A_{av}t_c}{lnN_c}$=1.

As a consequence, we get

$$b_c v t_c \leq (e^{b_c v t_c} - 1) \qquad (34)$$

A close observation of the inequality (34) reveals that any $b_c \leq 0$ also satisfies it

This result, in fact, motivates us to consider the negative critical Gom-pertz parameter.On the other hand, in the neighbourhood of the critical point $\frac{A_{av}t_c}{lnN_c} = 1$, it is necessary that $\frac{A_{av}t_c}{lnN_c} > 1$ if b<0 from (30).

Hence

$$-b_c v t_c \leq (e^{-b_c v t_c} - 1) \qquad (35)$$

Is valid for $N_0 = N_c$ provided $\frac{A_{av}t_c}{lnN_c} > 1$. Clearly $\frac{A_{av}t_c}{lnN_c} > 1$ holds when $t_c > \frac{lnN_0}{A_{av}}$ in (32). In case of (31) $\frac{A_{av}t_c}{lnN_c} > 1$ if $N_0 < N_c$ and this gives $t_c > \frac{lnN_0}{A_{av}} = t_c\frac{lnN_c}{lnN_0}$. We designate $t_m^* = t_c\frac{lnN_c}{lnN_0}$ as the actual maximum life span. Since t_m^* becomes very large numerically as $N_0 \longrightarrow 1$, we can choose any finite value greater than t_c as t_m^*. Fortunately, we shall not use the numerical value of t_m^* in our sequel.

Numerical experiments show that the transition from b_c to b_c is ex-tremely slow. The graphical illustration should enhance the understanding of this idea [19]. Further, it follows from (28) that as

$$b \longrightarrow 0 \; at \; N_0 = N_c, M_r \longrightarrow \frac{lnN_c}{t_c} = A_{av},$$

As b_c changes sign, from (27) we obtain

$$\frac{M_r}{-b_c v}\left(\frac{l_0}{l_r}\right)^{-b_c} = \frac{lnN_c}{e^{b_c v t_c} - 1},$$

Which gives

$$M_r = \frac{b_c v t_c}{e^{b_c v t_c}-1}\left(\frac{l_0}{l_r}\right)^{-b_c}\frac{lnN_c}{t_c} \geq \frac{lnN_c}{t_c}$$

Thus

$$M_r \geq \frac{lnN_c}{t_c}.$$

G.Exponential Allometry Survival function

In the neighbourhood of the critical point $\frac{A_{av}t_c}{lnN_c} = 1$, in view of (31) and (32)the survival function (23) takes the form

$$S(t) = \begin{cases} e^{-M_r\left(\frac{l_0}{l_r}\right)^{b}\frac{e^{bvt}-1}{bv}} & if\ t \le t_c (N \ge N_c) \\ e^{-M_r\left(\frac{l_0}{l_r}\right)^{-b}\frac{1-e^{bvt}}{bv}} & if\ \ t \ge t_c (N \le N_c) \end{cases} \quad (36)$$

Lemma: S(t) is continuous at $t = t_c$

Proof Indeed

$$\lim_{t\to t_{c-0}} S(t) = e^{-M_{rc}\left(\frac{l_0}{l_r}\right)^{b_c}\left(\frac{e^{b_c v t_c - 1}}{b_c v}\right)}$$

$$= e^{\frac{\ln N_c}{e^{b_c v t_c - 1}}\left(1 - e^{b_c v t_c}\right)}$$

$$= e^{\frac{\ln N_c}{e^{-b_c v t_c - 1}}\left(1 - e^{-b_c v t_c}\right)}$$

$$= e^{-M_{rc}\left(\frac{l_0}{l_r}\right)^{-b_c}\left(\frac{1 - e^{b_c v t_c}}{b_c v}\right)}$$

$$= \lim_{t\to t_{c+0}} S(t).$$

Now we extend the formula (33) to $-b_c$ for $N_0 = N_c$ satisfy (31) with $N \le N_c$ and (32) with $t_c \ge \frac{\ln N_0}{A_{av}}$

Upon substitution $b_c = -b_c$ into (33) we get

$$-b_c = \frac{A_{av}}{v} e^{\frac{\ln N_c}{1 - e^{-b_c v t_c}}} \int_{\frac{\ln N_c}{1 - e^{-b_c v t_c}}}^{0} \frac{e^{-z}}{z}\, dz. \quad (37)$$

As such the above integral is unbounded. The unboundedness results in by the substitution $t_c = \infty$ or $N_0 = 1$ with $N \le N_c$ into (33), when $b_c < 0$. But inequality (35) implies the existence of b_c.

To overcome this situation, we need to introduce limit age, t_{lim} of the critical population N_c.

Following Suematsu [32], the limit age can be defined as an age where the final member of the critical population N_c disappears. Stating mathematically,

$$S_{(tim)} = e^{M_r\left(\frac{l_0}{l_r}\right)^{-b}\left(\frac{1 - e^{-bvt_{tim}}}{b\ \ v}\right)} \le \frac{1}{N_0}$$

Clearly, if the population N_c at an age t t_c is less than unity, all the members under discussion must, in the statistical mean, vanish. Hence, the limit age can be identi ed with the minimum age, t_{lim}, that satisfies

$$S_{(tim)} \le \frac{1}{N_0}.$$

Taking into account the above arguments, substitute $N_0 = 0$ ($N_0 \le N_c$) into the upper limit of (33) when $b_c < 0$ to get

$$-b_c = \frac{A_{av}}{v} e^{-\frac{\ln N_c}{1 - e^{-b_c v t_c}}} \int_{\frac{-\ln N_c}{1 - e^{-b_c v t_c}}}^{0} \frac{e^{-z}}{z}\, dz. \quad (38)$$

$$= -\frac{A_{av}}{v} e^{-\frac{\ln N_c}{1 - e^{-b_c v t_c}}} E_i\left(\frac{\ln N_c}{1 - e^{-b_c v t_c}}\right),$$

where $E_i(x) = -\lim_{n\to+0}\left[\int_{-x}^{-\eta} \frac{e^{-z}}{z}\, dz + \int_{\eta}^{\infty} \frac{e^{-z}}{z}\, dz\right]$.

[x>0].

Clearly, (38) satisfies inequality (35). We could not obtain (35) directly from (38), since the integrand is unbounded at z = 0.

From equation (38), on account of 8.214. 2 in [33], we get

$$-b_c = -\frac{A_{av}}{v} e^{-\frac{\ln N_c}{1 - e^{-b_c v t_c}}} E_i\left(\frac{\ln N_c}{1 - e^{-b_c v t_c}}\right),$$

$$= -\frac{A_{av}}{v} e^{-\omega}\left[C + \ln\omega + \sum_{k=1}^{\infty} \frac{\omega^k}{k\, k!}\right].$$

Where C=0.577215, Euler's constant $\omega = \frac{\ln N_c}{1 - e^{b_c v t_c}}$

Next, we shall obtain a greatest lower bound for b_c. From the above equation, we get

$$-b_c \ge -\frac{A_{av}}{v} e^{-\omega}[C + \ln\omega + e^{\omega} - 1],$$

Since

$$\sum_{k=1}^{\infty} \frac{x^k}{k\, k!} < \sum_{k=1}^{\infty} \frac{x^k}{k!} < \sum_{k=1}^{\infty} \frac{x^k}{k!} = e^x - 1.$$

Further

$$-b_c \ge -\frac{A_{av}}{v} e^{-\omega}[C + \ln\omega] - A_{av}[1 - e^{-\omega}]$$

$$\ge -\frac{A_{av}}{v} e^{-\omega}[C + \ln\omega] - \frac{A_{av}}{v}$$

$$\ge -\frac{A_{av}}{v}[C + 1] \ge -\frac{A_{av}}{v} \max(\ln\omega, e^{-\omega})$$

$$= -\frac{A_{av}}{v}[C + 1] - 0.097\frac{A_{av}}{v}$$

$$= -\frac{A_{av}}{v}[1.097 + C]$$

On the other hand, from the inequality (35) we get

$$-b_c v t_c = e^{b_c v t_c - 1}$$

Combining, finally we get

$$-\frac{A_{av}}{v}[1.097 + C] \le -b_c v t_c = e^{-b_c v t_c - 1}$$

Summing up, we conclude that

Theorem 2: For every fixed A_{av}; t_m and N_0 satisfying either (31) with $N_0 \le N_c$ or (32) with $t_c \ge \frac{\ln N_0}{A_{av}}$, let I be the interval defined by

$$I = \left(-\frac{A_{av}}{v}[1.097 + C], 0\right)$$

where C = 0:577215, Euler's constant.

Suppose there exists a unique solution of (38) in I. Then it is necessary that. $\frac{A_{av} t_c}{\ln N_c} = 1$

Moreover, the following estimation is true

$$-b_c \geq -\frac{A_{av}}{v}[1.097 + C]$$

Remark 2. From [20], here we listed in Table I the numerical values of b_c for a comparison with that of positive b_c for $N_0 = N_c$.

Remark 3. From (27) it is easy to get the asymptotic formula of initial mortality rate using (38).

Remark 4. Kai Lorenzen [16] listed in Tables 1 and 2 the numerical values of each parameter. It is useful to compare our asymptotic formula [30] and the existence of critical and negative critical allometry parameter.

Table I

A_{av}/Year	$-b_c$	b_c
Pipepestrell bat 0.36	-0.602	0.244
European robin 0.62	-1.038	0.253
Lapwing 0.34	-0.569	0.177
Starling 0.52	-0.870	0.163
Common swift 0.18	-0.301	0.127
Herring gull 0.34	-0.569	0.180
Human 0.015	-0.025	0.024
Mouse 0.74	-1.239	0.888
Rat 0.64	-1.072	0.845
Japanese quail 0.35	-0.586	0.295

III. CONCLUSION

It is well known that the average mortality rate di ers between males and females. For instance, (Humans) 0.005 for males and 0.015 for females [19-20]. The issue of sex di erences is not addressed in this article and presents further complexities. The body size dependent relationships of mortality and longevity are examined for birds and eutherian mammals [W. A. Calder III]. Dif-ferences between mass exponents for maximum recorded longevity and survival times for fractions of original adult populations con rm the age dependence of mortality in both classes and a size dependency of population age distribution [16]. It is worth studying further the effect of body size on the mortality rate with sex differences.

ACKNOWLEDGEMENT

Heartfelt thanks to Mrs. Pandeeswari Asokan M.E., M.B.A., (P.hD) and Ms.Bhavatharani Asokan for typeset in word document.

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