# Infinite Beams on Elastic Foundation by using Meshfree Method

**H. Y. Kaundanyapure, P. J. Salunke, N. G. Gore**

***Abstract— The present studies emphasis the analysis of beam for elastic foundation using the Element Free Galerkine Method (EFGM). The attempt was made to provide a simple model for beams on elastic foundation using Mesh Free Technique, called as Element Free Galerkine Method which does not rely on the mesh. The EFGM presented in the study employs generalized Method of Least Square (MLS), which is used to construct shape function based on the set of nodes. The Discrete system equation are derived from the variation form of system equation. A FORTRAN and MATLAB program is developed and numerical example of finite and infinite beams on elastic foundation are presented. Numerical examples are provided to study the convergence and the efficiency of the method.***

***Index Terms— Elastic foundation, Element Free Galerkine Method (EFGM), beams on elastic foundation, Mesh Free Technique and Method of Least Square interpolation***

## I. INTRODUCTION

Highlight Recently, a new numerical method called mesh-free method is under development in the area of computational mechanics. There are different versions of the mesh-free method developed so far to analyse the stress and displacement in the solid. A Mesh-free method is a method used to establish system algebraic equations for the whole problem domain without the use of a predefined mesh for the domain discretization. Mesh-free methods represent the problem domain and the boundaries with sets of scattered nodes in the domain and on the boundaries. These sets of scattered nodes do not form meshes unlike the other numerical methods such as finite element method, finite difference method etc. Mesh-free method is a new numerical analysis method having excellent accuracy and rapid convergence.

In 1992, Moving Least Square approximation is used in a mesh free method (Galerkin Method) which is pioneered by [1] for solving partial differential equations. And they named that method as the Diffuse Element Method (DEM). After them, the method has been modified and refined by [2] and named as Element Free Galerkin (EFG) method. In this method, the moving least-squares interpolates were used to construct the trial and test functions for the variational principle (weak form) and weight functions.

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**Mr. H.Y. Kaundanyapure**, (M.E. student) Civil Engg. Dept., M.G.M‟s. College of Engineering & Tech., Kamothe, Navi Mumbai, India.

**Prof. P. J. Salunke**, (Guide) Civil Engg. Dept., M.G.M‟s. College of Engineering & Tech., Kamothe, Navi Mumbai, India.

**Prof. N. G. Gore**, (Guide) Civil Engg. Dept., M.G.M‟s. College of Engineering & Tech., Kamothe, Navi Mumbai, India.

Domain integration by Gauss quadrature in the Galerkin mesh-free methods adds considerable complexity to solution procedures. Direct nodal integration, on the other hand, leads to a numerical instability due to under integration and vanishing derivatives of shape functions at the nodes. Another new approach is developed in [3]. The author proposed a strain smoothing stabilization for nodal integration to eliminate spatial instability in nodal integration. For convergence, an integration constraint was introduced as a necessary condition for a linear exactness in the mesh-free Galerkin approximation.

The EFG method for analysing the beams having a central point load and uniformly distributed load and two end point loads, supported on elastic foundation is presented in [4]. A new mesh free method which is the point interpolation method based on radial basis function (RPIM) is given in [5].

The analysis of thin plate of complicated shape using a mesh-free method is presented in [6]. Method uses moving least-squares (MLS) interpolation to construct shape functions based on a set of nodes arbitrarily distributed in the analysis domain. Discrete system equations are derived from the variational form of system equation.

In the present work, the mesh-free method is used for soil-structure interaction problems and presented the formulation for finite and infinite beams on elastic foundation using Element Free Galerkin (EFG) method which is a one type of mesh-free method. The EFG method presented employs generalized moving least square approximation to generate the shape functions and the essential boundary conditions are enforced directly at each constraint boundary point. A parametric study is performed to investigate the effect of few selected parameters.

## II. FORMULATION OF THE PROBLEM

Let $u(x)$ be the function of the field variable defined in the domain Ω. The approximation of $u(x)$ at point $x$ is denoted $u$h($x$) which is given by MLS approximation as:

$$u^h(x) = \sum_{i=1}^{n} p_i(x)a_i(x) + \frac{d}{dx} p_i(x)a_i(x)$$

$$= p^T(x)a(x) + p_x^{\ T}(x)a(x) \qquad (1)$$

In the above equation, $p(x)$ is a vector of complete basis functions (usually polynomial) and $a(x)$ is a vector of coefficients given as:

$$p^T(x) = [1,\ x,\ y,\ z,\ xy,\ yz,\ zx, \ldots\ x^{k'},\ y^{k'},\ z^{k'}] \qquad (2)$$

$$a^T(x) = [a_1(x),\ a_2(x),\ a_3(x),\ \ldots\ a_n(x)] \qquad (3)$$

Where, $\mathbf{x}^T = [x \quad y \quad z]$

The unknown coefficients $a(x)$ are obtained by minimizing a weighted least square sum of the difference between local approximation, $u^h(x)$ and field function nodal parameters *u1* The weighted least square sum denoted by *J(x)* can be written in following quadratic form:

$$J(x)=\sum_{i=1}^{n} w_i(x)\left[p^T(x_i)a(x)-u_i\right]^2+w_i(x)\left[p_x^T(x_i)a(x)-\theta_i\right]^2 \quad (4)$$

Where, *u1* is the nodal parameter associated with node *i* at *x=xi* but these are not the nodal values of $u^h(x=xi)$ because $u^h(x)$ as an approximant not an interpolate.

$$\frac{dJ}{da}=0$$

$$=\sum_{i=1}^{n}(w_i(x)[p_i]^T u_i+w_i(x)\,[p_i^x]^T\theta_i) \quad (5)$$

[pi] =[1 xi xi2 xi3] and
[pix]=[0 1 2xi 3xi2]

Equation (5) can be written as,

$$A.a(x)=P^T W\{u\}+P_x^{\,T}W\{\theta\}$$

$$a(x)=A^{-1}P^T W\{u\}+A^{-1}P_x^{\,T}W\{\theta\} \quad (6)$$

From equation (1)

$$u=P^T(x)A^{-1}P^T W\{u\}+P_x^{\,T}(x)A^{-1}P_x^{\,T}W\{\theta\}$$

Consider

$$\phi_u=P^T A^{-1}P^T W$$

And

$$\phi_\theta=P_x^{\,T}A^{-1}P^T W \quad (7)$$

$$u=\phi_{u1}.u_1+\phi_{\theta 1}.\theta_1+\phi_{u2}.u_2+\phi_{\theta 2}.\theta_2+......... \quad (8)$$

Introducing $\gamma_u(x)=A^{-1}P^T W$ and differentiate it with respect to *x* after rearrangement

$$A\gamma_u(x)=P^T W \quad (9)$$

Now differentiate this $w.r.t$ to x

$$A\frac{d\gamma}{dx}=P^T\frac{dW}{dx}-\frac{dA}{dx}\gamma(x) \quad (10)$$

Again differentiating w.r.t. x we get

$$A\frac{d^2\gamma}{dx^2}=P^T\frac{d^2W}{dx^2}-(\frac{d^2A}{dx^2}\gamma(x))-2\frac{dA}{dx}\frac{d\gamma}{dx}$$

$$\phi_u=P^T(x)\gamma(x) \quad (11)$$

$$(12)$$

$$\frac{d\phi_u}{dx}=P^T(x)\frac{d\gamma}{dx}+P_x\gamma(x) \quad (13)$$

$$\frac{d^2\phi_u}{dx^2}=P^T(x)\frac{d^2\gamma}{dx^2}+2P^T{}_x(x)\frac{d\gamma}{dx}+P^T{}_{xx}(x)\gamma(x) \quad (14)$$

Similarly for rotation

$$\frac{d^2\phi_\theta}{dx^2}=P_x^{\,T}\frac{d^2\gamma_\theta}{dx^2}+2P^T{}_{xx}\frac{d\gamma_\theta}{dx}+P^T{}_{xxx}\gamma_\theta \quad (15)$$

First order and second order derivatives of matrix *A* can be computed using following expressions.

$$\frac{d^2A}{dx^2}=\sum_{i=1}^{n}\frac{d^2}{dx^2}W_i(x)\begin{Bmatrix}1\\x_i\\x_i^2\\x_i^3\end{Bmatrix}\{1\quad x_i\quad x_i^2\quad x_i^3\}+\sum_{i=1}^{n}\frac{d^2}{dx^2}W_i(x)\begin{Bmatrix}0\\1\\2x_i\\3x_i^2\end{Bmatrix}\{0\quad 1\quad 2x_i\quad 3x_i^2\} \quad (16)$$

Transformation matrix is given as, (16)

$$\Lambda=\begin{bmatrix}\phi_{u1}(x_1) & \phi_{\theta 1}(x_1) & \phi_{u2}(x_1) & \phi_{\theta 2}(x_1)...\\ \frac{d}{dx}\phi_{u1}(x_1) & \frac{d}{dx}\phi_{\theta 1}(x_1) & \frac{d}{dx}\phi_{u2}(x_1) & \frac{d}{dx}\phi_{\theta 2}(x_1)....\\ \phi_{u1}(x_2) & \phi_{\theta 1}(x_2) & \phi_{u2}(x_2) & \phi_{\theta 2}(x_2)...\\ \frac{d}{dx}\phi_{u1}(x_2) & \frac{d}{dx}\phi_{\theta 1}(x_2) & \frac{d}{dx}\phi_{u2}(x_2) & \frac{d}{dx}\phi_{\theta 2}(x_2)..\\ . & . & . & .\\ . & . & . & .\end{bmatrix}.=\begin{bmatrix}\varphi^T(x_1)\\ \frac{d}{dx}\varphi^T(x_1)\\ \varphi^T(x_2)\\ \frac{d}{dx}\varphi^T(x_2)\\ ...\\ ...\end{bmatrix} \quad (17)$$

A Governing equation for beam on an elastic foundation is given by the following 4th order differential equation is given below, over the domain 0 to L. where u is transverse displacement, EI is flexural rigidity, k is the foundation stiffness, and q is distributed load over the beam. The boundary conditions are given at the global boundary,

$$EI\frac{d^4u}{dx^4}+ku=q$$

$$=EI\int_0^L\left[\frac{d^2\varphi^T}{dx^2}\Lambda^{-1}\widehat{d}\right]^T\left[\frac{d^2\varphi^T}{dx^2}\Lambda^{-1}\widehat{d}\right]dx \quad (18)$$

The final equation can be written in the form *Kd=f*, where *K* is the stiffness matrix with

$$K=\Lambda^{-T}\begin{bmatrix}k_{11} & k_{12} & . & k_{1N}\\ k_{21} & k_{22} & . & k_{2N}\\ . & . & . & .\\ k_{N1} & k_{N2} & . & k_{NN}\end{bmatrix}\Lambda^{-1} \quad (19)$$

is the stiffness matrix with

$$k_{IJ}=\int_\Omega B_I^{\,T}EIB_J\,d\Omega+\int_\Omega \varphi_I^{\,T}k_s\varphi_J\,d\Omega \quad (20)$$

where,

$$B_I^{\,T}=\begin{bmatrix}\dfrac{d^2\phi_{uI}(x)}{dx^2}\\ \dfrac{d^2\phi_{\theta I}(x)}{dx^2}\end{bmatrix} \text{ and } \varphi_I^{\,T}=\begin{bmatrix}\phi_{uI}(x)\\ \phi_{\theta I}(x)\end{bmatrix} \quad (21)$$

Load vector is given as follow:

$$\widehat{f}=\Lambda^{-T}f$$

Where.


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$$f = \int_0^L \varphi_I{}^T q(x)dx + [\varphi_I{}^T EIu,_{xxx}]_{x=0} - \left[\frac{d\varphi_I{}^T}{dx} EIu,_{xx}\right]_{x=0} \quad (22)$$

Displacement is given as follow:

$$K\hat{d} = \hat{f}$$

$$\hat{d} = K^{-1}\hat{f}$$

If point load $P$ is acting at centre of beam of infinite length as shown in Fig.1 then two types of boundary conditions should be considered as follows:
Slope at the location of point load, $\theta = 0$
Shear force at the location of point load, $V= -0.5P$

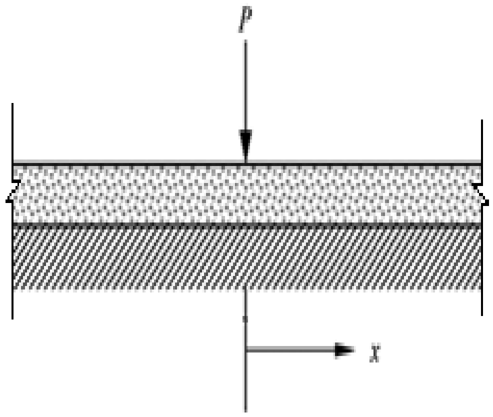


**Fig 1 Beams on elastic foundation with central point load**

## III. RESULT

A computer program based on the formulation is developed in FORTRAN. A parametric study is carried out to study the effect of parameters such as number of field nodes, length of beam & modulus of subgrade reaction (*ks*) on the response of beams on elastic foundation. A beam of infinite length and having flexural rigidity of $EI$= 20000 kN/m$^2$ subjected to central point load ($P$ =1000 kN) is considered. However nodes are considered upto distance $L$ of 40 m from centre. soil modulus *ks* =2000 kN/m$^3$ was considered. Number of field nodes was varied as 6, 11, 16 and 21 in the present study to examine their effect on prediction. Results obtained from the study are represented in the graphical form Fig. 2 and Fig.3 in the form of deflection curve along the length of beam for various conditions.

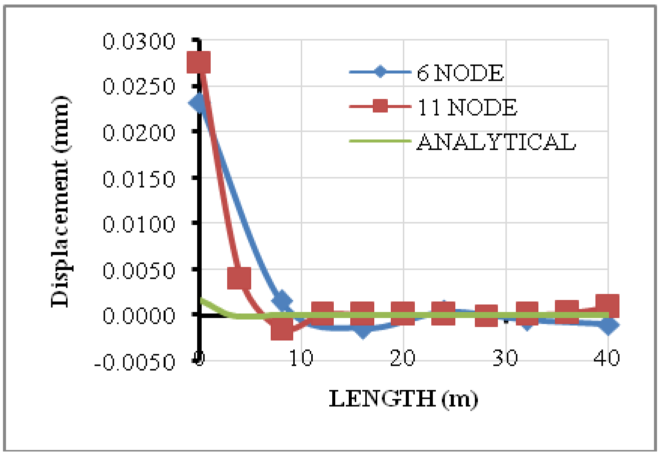


**Fig. 2 Displacement profile for L=40m and ks=2000 kN/m3**

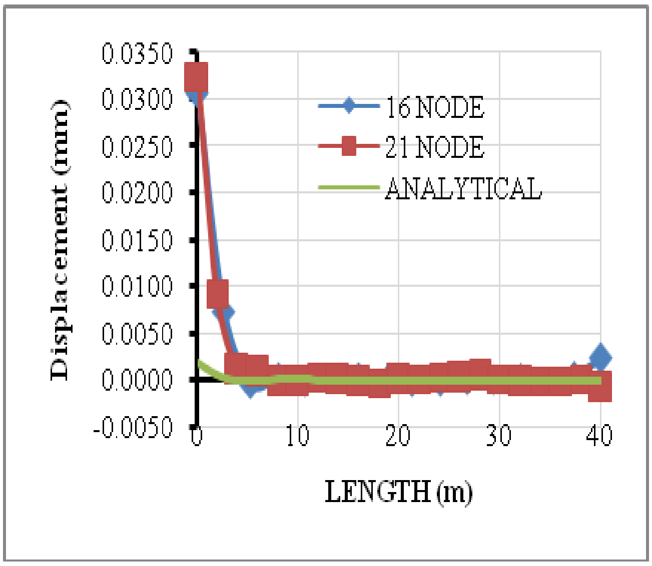


**Fig. 3 Displacement profile for L=40m and ks=2000 kN/m3**

From Fig. 2 and Fig. 3, it can be seen that the displacements are much nearer to the analytical solution with 16 and 21 nodes as compare to 6 and 11 nodes except at centre of the beam where central deflection with 6 nodes is closer to analytical solution. Not much difference is observed between the displacement profiles for 16 and 21 nodes. For the case of $L$=40m and *ks* = 2000 kN/m$^3$ when the beam is descritized in 11 nodes the observed displacement is about 19.4 % more than the 6 nodes case (Fig. 2) and the displacement of 21 node case is 5.24 % more than the 16 nodes case (Fig. 3). So, it can be said that when the beam is descritized using particular number of node, the difference between displacements obtained by EFG and analytical method is minimum.

## IV. CONCLUSION

- A new Mesh Free method is developed for solving soil structure interaction problems such as infinite beams on elastic foundation.
- Displacements are taken as field variables. Unlike the finite element method, the Mesh Frees method requires no structured mesh, since only a scattered set of nodal points is required in the domain of interest. There is no need for fixed connectivity between the nodes.
- The Element Free Galerkin Method is proposed for the analysis of beams on elastic foundation. In this study, computer codes using FORTRAN and MATLAB has been developed to analyze the infinite beams on elastic foundation with central point load by EFG method and to examine the accuracy and convergence of the mesh less method in finding the displacement, slope and bending moment along the length of beams.
- The results obtained by EFG method are compared with analytical solution. An average agreement is observed between the results of the EFGM method and the analytical solutions.
- In the case of the analysis with only displacement parameter, results converge towards the analytical solution with increasing the field nodes.


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- Since mesh generation of complex structures can be a far more time-consuming and costly effort than the solution of a discrete set of equations, the current meshless method provides an attractive alternative to the finite element method for solving soil structure interaction problems such as beams on elastic foundation

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